Privileged information exacerbates market volatility *†

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Abstract

We study how asymmetric information affects market volatility in a linear setup where the outcome is determined by forecasts about this same outcome. The unique rational expectations equilibrium will be stable when it is the only rationalizable solution. The literature has established that stability is obtained when the sensitivity of the outcome to agents' forecasts is less than 1, provided that this sensitivity is common knowledge. Relaxing this common knowledge assumption, instability is obtained when the proportion of agents informed of the sensitivity is large, and the uninformed agents believe it is possible that the sensitivity is greater than 1.

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1 Introduction

One puzzle in the theory of economic fluctuations concerns the high degree of market volatility. Market volatility appears to be partially disconnected from economic fundamentals and is often found to be excessive, especially in financial markets, including stock and currency markets, and in agricultural or energy markets. Much effort has been put into understanding the causes of this abnormal volatility. For instance, a recent avenue of study argues that our lack of comprehension of business cycle fluctuations may be due to shocks observed by economic agents, e.g., news shocks about the true underlying economic fundamentals or forecasts shocks.[1] This paper examines how market volatility is related to the way in which the news is spread throughout the economy. It shows that a departure, even a very small one, from a situation in which everyone is aware of the true underlying economic fundamentals is likely to generate instability by favoring expectations coordination failures.

Coordination problems typically arise when an agent's decision depends on his forecast regarding some aggregate outcome, which itself is determined by the decisions of all economic agents. In this configuration, the market exhibits a beauty contest in which every agent tries to predict the behavior of others. This self-referential aspect is relevant in financial markets. The attempt to predict the behavior of others is crucial in the model of currency attacks analyzed by Morris and Shin (1998), in which every trader decides whether to attack the currency, and the payoff for attacking depends on the proportion of traders who decide to attack. Similarly, in the agricultural market studied by Guesnerie (1992), every farmer chooses his level of corn production before knowing the market clearing price of corn, and this price in turn depends on the aggregate supply produced by all the farmers. A key aspect of the beauty contest component is that agents face strategic uncertainty: they are *a priori* uncertain about the behavior of others. The literature has suggested circumstances in which agents can succeed in guessing others' behavior. A simple and general lesson illustrated in this area is the crucial role played by a single parameter: the sensitivity of the actual aggregate outcome (e.g., the proportion of agents who attack or the aggregate production of corn) to agents' beliefs about it. A low 'sensitivity to beliefs' makes it more likely that agents will correctly predict others' behavior. When the sensitivity becomes high enough, correct prediction is no longer likely. With high sensitivity, strategic uncertainty persists, and market volatility results from coordination failures.

The sensitivity to beliefs is part of economic fundamentals. In the corn market, this sensitivity depends on the price elasticities of aggregate supply and demand. All coordination issues would vanish in the polar case where there is an inelastic supply. In such a case, the aggregate supply would be unrelated to price forecasts, so each individual farmer actually would not need to predict the behavior of the other farmers. Through continuity, coordination issues will be solved when the price elasticity of supply is low in comparison to the price

[1] Business cycle models with news shocks stress amplification mechanisms through which news disclosures imply volatility (Beaudry and Portier, 2006; Beaudry and Portier 2007; Jaimovich and Rebelo, 2009). A voluminous literature studies how announcements influence exchange rate volatility (Andersen, Bollerslev and Diebold, 2010), macroeconomic volatility (DeGennaro and Shrieves, 1997; Jansen and Haan, 2007) and the following correction (De Bondt and Thaler, 1987).

elasticity of demand. In the currency attacks model, the sensitivity to beliefs relies on the precision and correlation of private signals received by traders about foreign exchange reserves. The sensitivity is low when private signals are strongly correlated; if these signals are perfectly correlated, traders are able to deduce the signals received by others based upon their own signal. Information may then remain imperfect, but it is no longer asymmetric.

In the literature, the sensitivity to beliefs is assumed to be common knowledge. The main innovation of the present paper is to relax this assumption. It shows that expectations coordination fails as soon as some agents accept the possibility that the sensitivity to beliefs is large in some state of nature. If, as seems plausible, this requirement is met in an uncertain world, instability resulting from the inability to pin down expectational behavior becomes the rule, rather than the exception.

When one relaxes common knowledge assumptions about economic fundamentals, both strategic uncertainty and fundamental uncertainty are significant. The interplay between these two types of uncertainty is usually captured by the Bayesian Nash equilibrium. In such an equilibrium, an agent generally ignores the actions taken by others, but all the agents know how private information about fundamentals relates to individual decisions. This scenario implies that strategic uncertainty would vanish in the absence of private information. For this reason, it is difficult to ascertain the specific role played by strategic uncertainty. The role of strategic uncertainty becomes clearer in the context of the weaker solution concept of rationalizable solutions. By definition, the set of rationalizable expectations (strategies) comprises all individual expectations that can be justified whenever expectations of others belong to the set. The set of rationalizable solutions is the largest self-fulfilling set of expectations, and although it comprises the Bayesian Nash equilibrium, it may not reduce to it. In the absence of such a reduction, some uncertainty remains about agents' behavior. Knowing the private information of an agent does not *a priori* allow other agents to infer his behavior. Considering rationalizable solutions instead of the Bayesian Nash equilibrium makes strategic uncertainty possible even in the absence of fundamental uncertainty.

When all agents know the true level of sensitivity to beliefs, a low sensitivity ensures that the set of rationalizable solutions reduces to the Bayesian Nash equilibrium, both in the eductive reasoning advocated by Guesnerie (1992) and in the global game approach used by Morris and Shin (1998). When some agents are unaware of the true sensitivity, all the possible values of the sensitivity are significant to the elimination of non-best response strategies. This reflects a contagion-like argument: what happens in a given state of nature is influenced by all the other states. If an agent ignores the true state, then obviously, he cannot condition his expectations to the actual state and must, therefore, form an expectation about the behavior of others in any possible state of nature. Other agents, to understand the behavior of any uninformed agents, must determine how they, themselves, would behave in every possible state of nature, even if they know the true one. Everyone therefore accounts for all the states to infer the behavior of others.

In our benchmark framework, the market outcome is determined by the expectations formed about it. It is a reduced-form model intended to focus on the beauty contest component. There is a linear relation between the market outcome and agents' expectations. Its slope measures the sensitivity to beliefs, and unlike the models in the earlier literature, it differs across states of nature. The linearity assumption implies that, generically, there is a unique Bayesian Nash equilibrium. This equilibrium coincides with the unique rational expectations equilibrium of the economy. Information about the sensitivity to beliefs is possibly asymmetric; some agents may be perfectly informed, whereas the others have no information. A change in the proportion of informed agents is used as a proxy for the overall precision of information. It provides insights into the effects of public macroeconomic announcements on stability: should a Bank with a stabilization purpose be transparent and announce its policy rule, or should it conceal information about economic fundamentals from uninformed agents ? If information is released, should the Bank prefer a partial revelation to only some of the uninformed, i.e., maintain privileged information, or identically inform all the agents ?

This paper highlights two different effects of changes to the information structure. The first one considers the introduction of a few uninformed agents into a market where all the agents previously had been informed. The paper demonstrates that this change can never narrow the set of rationalizable solutions. In this sense, it is necessarily 'destabilizing.' The typical situation is one where the set of rationalizable solutions initially only consists of the Bayesian Nash equilibrium; whereas, there is a continuum of rationalizable solutions once the introduction of a few uninformed agents occurs. This discontinuity when one departs from the symmetric perfect information case is a straightforward consequence of the contagion property, which is not be effective in the initial situation but becomes so in the presence of uninformed agents. This discontinuity is obtained as soon as the sensitivity to beliefs is not common knowledge and some agents believe that the sensitivity may be high.

This property is similar to the instability results derived by studies in the adaptive learning literature. For instance, according to the 'uncertainty principle' advocated by Grandmont (1998), adaptive learning dynamics diverge from rational expectations equilibria when agents are uncertain about the stability of the system, and are ready to extrapolate a large range of regularities, including divergent trends. This principle may explain why the reaction of the economy to news sometimes appears to be disproportionate to the news content (Cutler, Poterba and Summers, 1989; Allen and Gale, 2007). This property is also reminiscent of Morris and Shin (1998): the theoretical insights obtained in the symmetric perfect information case are not robust when there are small changes in the structure of information available to agents. In Morris and Shin (1998), the discontinuity refers to the number of equilibria. In our linear setup, there is always a unique equilibrium, and the discontinuity refers to the stability property of this equilibrium, i.e., whether it is the unique rationalizable solution. From a policy viewpoint, it is perturbing point that weaker common knowledge assumptions yield a unique equilibrium in Morris and Shin (1998); whereas, they yield a multiplicity of rationalizable solutions in our setup. The loss of common knowledge about a low sensitivity to beliefs makes a unique equilibrium a less plausible solution.

The second result more specifically considers the role played by uninformed agents in the process of elimination of dominated expectations. In an initial situation where there are uninformed agents, this paper considers how the disclosure of the true sensitivity to only some of these agents, so that there are still uninformed agents in the final situation, affects the set of rationalizable outcomes.

This paper shows that an increase in the proportion of informed agents cannot yield a narrower set of rationalizable solutions. Indeed, this set only comprises the Bayesian Nash equilibrium if and only if the proportion of informed agents is below some threshold proportion. In the presence of asymmetric information, a smaller proportion of informed agents is 'stabilizing.' This conclusion hinges on the inertia of the behavior of an uninformed agent. When such an agent expects others to change their behavior in some state of nature, his reaction to this expectation will be dampened because he is uncertain whether this state will occur. This inertia enables other agents to more easily understand his behavior. Consequently, accurate predictions are more likely.

This second result conforms to much of the recent literature concerned with macroeconomic stabilization issues (Woodford, 2003; Hellwig, 2008; and Nimark, 2008). These papers show that informational asymmetries may imply a greater persistence of equilibrium fluctuations. In the presence of informational asymmetries, an agent is not able to assess exactly how a shock to fundamentals influences others' decisions. Thus, as far as his optimal decision depends on others, his Bayesian Nash equilibrium behavior implies a slow reaction to his private information. Our paper shows that this logic extends to out-of equilibrium behavior. It also matters in the process of elimination of dominated strategies.

The paper is organized as follows. The benchmark setup is presented in Section 2. It encompasses Guesnerie (1992) and a linear (local) version of Morris and Shin (1998). The process of iterated elimination of dominated strategies in the case of complete information is briefly described in Section 3. In Section 4, the analysis is extended to the case of asymmetric information, and the main results are given. Section 5 discusses possible extensions of the work to informational efficiency, higher order uncertainty, and extraneous uncertainty of the sunspot type.

2 The framework

We consider a stylized model in which agents face a beauty contest issue. There is a continuum of infinitesimal agents $i \in [0,1]$ who simultaneously form forecasts (p_i^e) about the 'price.' These forecasts then determine the actual price. The uncertainty about fundamentals is represented by Ω states of nature indexed by ω, $\omega = 1, \ldots, \Omega$. In state ω, the actual price $p(\omega)$ is governed by the linear temporary equilibrium relation

$$p(\omega) = \phi(\omega) \int_0^1 p_i^e di + \eta(\omega) . \tag{1}$$

Fundamentals in state ω are summarized by a pair $(\phi(\omega), \eta(\omega))$. The real number $\phi(\omega)$ is the sensitivity to beliefs (it measures the sensitivity of the actual price to agents' forecasts) and $\eta(\omega)$ is a scale factor. In the sequel, we assume that the model exhibits strategic complementarity, i.e., $\phi(\omega) > 0$ for every ω. This is the case in Morris and Shin (1998, 2002), where $\phi(\omega) = \phi \in (0,1)$. This is primarily a matter of presentation. Our analysis applies in the presence of strategic substitutability, i.e., $\phi(\omega) < 0$ for every ω, as in Guesnerie (1992). This analysis would not extend to the case where the signs of the sensitivity to beliefs differ across states of nature; however, this may not be the most economically relevant configuration.

Example 1. The Muth model (Guesnerie, 1992). There is a continuum of farmers $i \in [0,1]$ who produce corn. Each farmer chooses his crop one period before observing the corn price. The cost of producing q units of corn is q^2/σ to each farmer, with $\sigma > 0$. Thus, when farmer i expects the price p_i^e, his expected profit is $p_i^e q - q^2/\sigma$. Profit maximization yields $q = \sigma p_i^e$. The actual price clears the market. The aggregate demand is $b - ap$. Aggregate supply equals aggregate demand when

$$\sigma \int p_i^e di = -ap + b,$$

which fits (1), with $\phi(\omega) = \phi = -\sigma/a$. In this example, the sensitivity is the same in every state. The sensitivity would vary across states of nature with uncertain aggregate demand, e.g., $b(\omega) - a(\omega)p$ in state ω $(a(\omega), b(\omega) > 0)$.

Example 2. Investment game. This setup is similar to the currency attack model (Morris and Shin, 1998). There is a continuum of traders $i \in [0,1]$. Each one must decide whether he will invest or not. When he invests, his payoff is $\theta + \ell - 1$, where θ is an unknown parameter, which stands for the intrinsic value of the investment, ℓ is the proportion of traders who invest, and the investment cost is 1. When trader i receives a private signal x_i on θ, his expectation $E(\theta \mid x_i)$ is assumed to be equal to x_i. In a 'switching' strategy, an agent i invests if and only if his private signal is above some cutoff point k_i. Let $F(k_j \mid x_i)$ be the probability that another agent j receives a signal lower than k_j given that agent i has received the signal x_i. Agent i believes that agent j invests with probability $1 - F(k_j \mid x_i)$. The proportion of investors expected by agent i is therefore

$$1 - \int_0^1 F(k_j \mid x_i) dj.$$

If agent i does not invest, his payoff is normalized to 0. Thus, the *ex ante* payoff of agent i is

$$\int_{k_i}^{+\infty} \left(x - \int_0^1 F(k_j \mid x) dj \right) dF(x),$$

where $F(x)$ stands for the unconditional distribution of private signals. The optimal (interior) threshold of agent i satisfies

$$k_i - \int_0^1 F(k_j \mid k_i) \, dj = 0.$$

This equation represents his best response function to a profile (k_j) of cutoffs chosen by all the other agents. The function is linear when private signals are uniformly distributed. More generally, in an arbitrarily small neighborhood of a symmetric equilibrium (where $k_i = k_j = k^*$),

$$k_i - k^* = \frac{F'_{k_j}(k^* \mid k^*)}{1 - F'_{k_i}(k^* \mid k^*)} \int_0^1 (k_j - k^*) dj, \quad (2)$$

where $F'_{k_j} \geq 0$ and F'_{k_i} are the partial derivatives of $F(k_j \mid k_i)$. Equation (2) fits (1) with $\phi(\omega) = \phi$. This parameter is positive whenever private signals are positively correlated. It can be either less than or greater than 1. With the Gaussian specification used by Morris and Shin (2000),

$$F'_{k_i}(k_j \mid k_i) = -F'_{k_j}(k_j \mid k_i)\frac{\operatorname{cov}(k_i, k_j)}{\operatorname{var}(k_j)}.$$

Hence, the sensitivity to beliefs is greater (resp. lower) than 1 when the ratio $\operatorname{cov}(k_i, k_j)/\operatorname{var}(k_j)$ is small (resp. large) enough, i.e., private signals are weakly (resp. strongly) correlated. In Morris and Shin's setup, the sensitivity to beliefs is constant. However, it would vary if the distributions $F(\cdot, \omega)$ of private signals were different across states of nature, or in the case of a random population size of potential investors. Such an example is detailed in the appendix.

Remark 1. These examples implicitly assume that all the agents involved are identical, in the sense that the influence of an agent on the actual price is the same for every agent ($\phi(\omega)$ does not depend on i). This assumption is made w.l.o.g. as long as the influence of an agent on the price is not correlated with his information and his expectations.

3 Complete information

In (1), the individual price forecasts implicitly depend on agents' information. This section focuses on the case in which it is commonly known that the state of nature is ω. This configuration is examined by Guesnerie (1992) and Morris and Shin (1998). Price forecasts are made conditionally on ω, $p_i^e = p_i^e(\omega)$ in (1). A rational expectations equilibrium (REE) is a price $p^*(\omega)$ such that $p_i^e(\omega) = p^*(\omega)$ for all i, that is, a price $p^*(\omega)$ such that $p^*(\omega) = \phi(\omega)\, p^*(\omega) + \eta(\omega)$. There is a unique REE as soon as $\phi(\omega) \neq 1$.

The REE can be viewed as the Nash equilibrium of a strategic 'guessing' game in which the strategy of agent j is a forecast $p_j^e(\omega)$. Each agent's objective is to minimize his squared forecast error $(p(\omega) - p_j^e(\omega))^2$, and $p(\omega)$ is determined by (1). Indeed, in this game, the best-response forecast of agent j to a profile $(p_i^e(\omega))$ of others' forecasts is

$$p_j^e(\omega) = \phi(\omega) \int_0^1 p_i^e di(\omega) + \eta(\omega). \tag{3}$$

Through this interpretation, every agent expects $p^*(\omega)$ because each believes that all the others expect $p^*(\omega)$. This (second order) belief is justified by higher order beliefs such that all the agents believe that all the rest expect $p^*(\omega)$. The price $p^*(\omega)$ is the only one consistent with the common knowledge (CK) of every agent expecting it.

Following Guesnerie (1992), this justification suggests an assessment of the REE relying on a weaker assumption than CK of $p_i^e(\omega) = p^*(\omega)$ for all i. Assume instead that it is CK that the actual price $p(\omega)$ belongs to some set $P^0 = \left[p^0_{\inf}, p^0_{\sup}\right]$ which comprises $p^*(\omega)$ but does not necessarily reduce to this price. From this assumption, it is CK that $p_i^e(\omega) \in P^0$ for all i. Then appealing to (1), all the agents can infer that the actual price will be in the set $P^1(\omega) = R_\omega(P^0)$ where the map R_ω is defined by

$$R_\omega(P) \equiv [\phi(\omega)P + \eta(\omega)] \cap P,$$

where P is any subset of prices. The actual price is determined by (1), provided that it is in P^0. Otherwise, it is the appropriate bound of P^0, either $p^0_{\inf}$ (if the price given by (1) is less than $p^0_{\inf}$) or $p^0_{\sup}$ (if the price is greater than $p^0_{\inf}$).

One defines a sequence of sets $P^\tau(\omega)$ along the same lines by $P^\tau(\omega) = R_\omega(P^{\tau-1}(\omega))$. It follows that if it is CK that $p(\omega) \in P^{\tau-1}(\omega)$, then it is CK that $p(\omega) \in P^\tau(\omega) = R_\omega(P^{\tau-1}(\omega))$. Then, the set of prices consistent with the common knowledge assumptions is the limit set

$$P^\infty(\omega) = \cap_{\tau \geq 0} P^\tau(\omega).$$

This limit set is properly defined since the sequence $P^\tau(\omega)$ is decreasing. The limit set is the set of rationalizable price forecasts of the 'guessing' game (where forecasts are *a priori* restricted to P^0).

The equilibrium is 'stable' when $P^\infty(\omega) = \{p^*(\omega)\}$. Otherwise, the REE is 'unstable.' Here, every price in P^0 is rationalizable when the REE is unstable. A necessary and sufficient condition for stability has been given by Guesnerie (1992).

Proposition 1. *The REE is stable if and only if* $\phi(\omega) < 1$.

This proposition provides a benchmark for our analysis of the asymmetric information case. Stability is obtained when the economic system is not too sensitive to forecasts in (1), or equivalently agents' forecasts are not too sensitive to others' forecasts in (3).

4 Imperfect asymmetric information

We now assume that there are only α $(0 \leq \alpha < 1)$ 'informed' agents who observe the underlying state of nature ω before choosing their price forecasts. The $(1-\alpha)$ remaining agents have no information about the true state of nature at that time. These 'uninformed' agents have common prior beliefs: they all believe that state ω occurs with probability $\pi(\omega)$. This homogeneity assumption will be relaxed in Section 5.2. Here, all the agents, both informed and uninformed agents, try to correctly predict the actual price. The REE is said to be 'stable' when they succeed in predicting the price; otherwise, it is 'unstable.'

Unlike the complete information case, the presence of uninformed agents implies that the stability of the REE necessarily involves all the states, not only the actual one. Indeed, uninformed agents always have to determine the prices in all possible states. The price in a given state ω depends on the expectations of the prices in any state, not only on the expectations of the price in state ω. It follows that the price in state ω is an equilibrium price only when agents correctly predict the price in any state, not only the price in state ω.

Following this argument, a REE is a vector of self-fulfilling prices $(p^*(1), \ldots, p^*(\Omega))$, that is, a vector of prices such that

$$p^*(\omega) = \phi(\omega)\left(\alpha p^*(\omega) + (1-\alpha)\sum_w \pi(w)p^*(w)\right) + \eta(\omega) \tag{4}$$

for any ω. The REE coincides with the Nash equilibrium of an amended 'guessing' game in which agents try to minimize their own forecast errors. This Bayesian game is as follows. First, the true state ω is observed only by the informed agents $i \in [0, \alpha]$. Then, all the agents simultaneously choose their forecasts. The strategy of agent i is a price forecast conditional on his information. If i is informed, his strategy is a vector of price forecasts $(p_i^e(1), ..., p_i^e(\Omega))$, where $p_i^e(\omega)$ is the price expected by i to arise in state ω. If i is uninformed, then his strategy merely consists of a single price forecast p_i^e independent of ω. The aggregate price forecast in state ω is therefore

$$\int_0^\alpha p_i^e(\omega)\, di + \int_\alpha^1 p_i^e di.$$

Finally, the actual price $p(\omega)$ is determined by the aggregate price forecast according to the map

$$p(\omega) = \phi(\omega) \left(\int_0^\alpha p_i^e(\omega)\, di + \int_\alpha^1 p_i^e di \right) + \eta(\omega)\,. \tag{5}$$

4.1 Stability of the equilibrium

Assume CK that the price *a priori* belongs to some interval P^0, which includes the equilibrium prices $p^*(\omega)$ for every ω. Every agent thus knows that all the other agents expect the price to be in P^0, and, consequently, each one understands that the aggregate price forecast is in P^0 in any state of nature. Hence, every agent concludes that the price in state ω belongs to the set $P^1(\omega) = R_\omega\left(P^0\right)$, which is included in P^0 and may coincide with P^0. When $P^1(\omega) \subsetneq P^0$, agents have succeeded in eliminating some price forecasts.

Iterating this process yields the CK restriction that the price in state ω is in some set $P^{\tau-1}(\omega)$ after $\tau - 1$ steps. At step τ, every agent knows that all the others expect the price in state ω to be in $P^{\tau-1}(\omega)$. Every agent understands that the price forecast in state ω of an informed agent is in $P^{\tau-1}(\omega)$, and that the price forecast of an uninformed agent is in $\sum_w \pi(w) P^{\tau-1}(w)$. All agents conclude that the price in state ω belongs to

$$P^\tau(\omega) = R_\omega \left(\alpha P^{\tau-1}(\omega) + (1-\alpha) \sum_w \pi(w) P^{\tau-1}(w) \right). \tag{6}$$

The relation (6) defines a sequence of intervals $(P^\tau(\omega), \tau \geq 0)$ for every ω. These sequences are decreasing and converge to limit sets $P^\infty(\omega)$. The REE is 'stable' whenever $P^\infty(\omega) = \{p^*(\omega)\}$ for every ω. Otherwise, it is 'unstable.'

Remark 2. As in Section 3, this definition has a game-theoretical counterpart in terms of rationalizable solutions (Bernheim 1984, Pearce 1984). At step τ, if the strategy set is restricted to $\times_\omega P^{\tau-1}(\omega)$ for an informed agent, and to $\sum_w \pi(w) P^{\tau-1}(w)$ for an uninformed agent, then the best-response of agent i is a strategy in $\times_\omega P^\tau(\omega)$ when he is informed, and in $\sum_w \pi(w) P^\tau(w)$ when he is uninformed. The limit sets $P^\infty(\omega)$ are the rationalizable price forecasts of the 'guessing' game: $P^\infty(1) \times \cdots \times P^\infty(\Omega)$ is the set of rationalizable price forecasts of an informed agent, and $\sum_w \pi(w) P^\infty(w)$ is the set of rationalizable price forecasts of an uninformed one. Stability of the REE is equivalent to the uniqueness of the rationalizable price forecast, which then reduces to the REE prices.

The following result presents the properties of the set of rationalizable prices when the REE is unstable.

Proposition 2. *Consider an unstable REE.*

1. *For every ω, $\{p^*(\omega)\} \subsetneq P^\infty(\omega)$: for every ω, the set $P^\infty(\omega)$ of rationalizable prices in state ω includes but differs from $\{p^*(\omega)\}$.*
2. *For every ω such that $\alpha\phi(\omega) > 1$, $P^\infty(\omega) = P^0$.*
3. *For every ω such that $\phi(\omega) < 1$, $P^\infty(\omega) \subsetneq P^0$, and $P^\infty(\omega)$ decreases in P^0: if $P^0 \subsetneq \tilde{P}^0$, then the limit sets $P^\infty(\omega)$ and $\tilde{P}^\infty(\omega)$ associated with the initial restrictions P^0 and $\tilde{P}^0$ are such that $P^\infty(\omega) \subsetneq \tilde{P}^\infty(\omega)$.*

The first item of this Proposition is a formal statement of the 'contagion' property. It shows that no price $p^*(\omega)$ can be guessed in the case of REE instability, even in a state ω where $\phi(\omega) < 1$. Indeed, uninformed agents cannot select a single price forecast when the REE is unstable. This situation implies that, in every state, agents cannot settle upon the aggregate price forecast. Therefore, the actual price, which is determined by the aggregate price forecast, cannot be uniquely determined.

When the equilibrium is unstable, some 'coordination' volatility occurs in all the states at the outcome of the process of elimination of non-best response strategies. The magnitude of this volatility can be measured in state ω by the size of the interval $P^\infty(\omega)$ of rationalizable prices. Volatility is dampened when $P^\infty(\omega)$ is a narrow interval around the REE price $p^*(\omega)$. The second and the third items of Proposition 2 characterize how the residual volatility depends on economic fundamentals. They show that a low sensitivity to beliefs $\phi(\omega)$ plays a role reminiscent of that in the complete information case. A low sensitivity favors a narrow set $P^\infty(\omega)$ of rationalizable prices in state ω. In the contrary case, in a state where $\phi(\omega)$ is large enough, the iterative process (6) provides no additional information: $P^\infty(\omega) = P^0$. These two items also show how the magnitude of this volatility depends on the initial assumption made about the relevant prices: a narrower prior set P^0 yields a narrower set $P^\infty(\omega)$ of rationalizable prices at the outcome of (6).

Thus far, we have focused on the description of an unstable REE. The next result yields conditions for stability of the REE.

Proposition 3. *Assume that $\phi(\omega) > 0$ for any ω. Let $0 \leq \alpha \leq 1$.*

1. *If $\alpha\phi(\omega) > 1$ for some ω, then the REE is unstable.*

2. If $\alpha\phi(\omega) < 1$ for every ω, then the REE is stable if and only if

$$\sum_{w=1}^{\Omega} \pi(w) \frac{(1-\alpha)\phi(w)}{1-\alpha\phi(w)} < 1. \tag{7}$$

The system (6) is a first-order linear recursive system. The REE is stable if and only if the spectral radius of the square matrix governing the dynamics (6) is less than 1. This yields the conditions given in Proposition 3.

The first item of this Proposition states that the REE is stable in (6) only if $\alpha\phi(\omega) < 1$ for every ω. This inequality would also govern stability of the REE in state ω in a complete information setup involving α informed agents only. This fact suggests one should interpret this inequality by referring to a virtual restricted coordination problem, which abstracts from the difficulties caused by uninformed agents. Namely, if informed agents know that the forecast of uninformed agents is fixed at $\bar{p}^* \equiv \sum_w \pi(w)p^*(w)$, then the actual price in state ω is

$$p(\omega) = \phi(\omega) \int_0^{\alpha} p_i^e(\omega)di + \tilde{\eta}(\omega)$$

where $\tilde{\eta}(\omega) = (1-\alpha)\phi(\omega)\bar{p}^* + \eta(\omega)$. This virtual restricted setup is formally equivalent to the complete information case discussed in the previous section (with a mass α of agents only). Hence, by Proposition 1, the REE is unstable when $\alpha\phi(\omega) > 1$. In this configuration, informed agents cannot correctly predict the price in state ω, even though, they know that uninformed agents expect the REE prices. It follows that in the true unrestricted setup, no agent (neither informed nor uninformed) succeeds in predicting the price in such a state.

4.2 Stability and precision of information

In Proposition 3, the effects of the level of the sensitivity to beliefs on the stability of the REE are qualitatively similar to those obtained under complete information. A rise in $\phi(\omega)$ is necessarily destabilizing: both $\alpha\phi(\omega)$ in the first item of Proposition 3 and the LHS of inequality (7) in the second item are increasing in any $\phi(\omega)$.

A novel feature, however, concerns the dispersion of the sensitivity to beliefs. Because the left-hand side (LHS) of (7) is a convex function of $\phi(\omega)$, a mean-preserving spread of the distribution of $\phi(\omega)$ increases this LHS. A more dispersed distribution of $\phi(\omega)$ makes an unstable REE more likely. Intuitively, the information of informed agents improves when the dispersion of sensitivity rises. This property, therefore, suggests that more precise information will be detrimental to stability. This intuition is confirmed by the following Corollary to Proposition 3.

Corollary 4. *Let $\phi(\omega) > 0$ for all ω. Then, there is a unique threshold proportion α^*, $0 \le \alpha^* \le 1$, of informed agents such that stability of the REE is obtained if and only if $\alpha < \alpha^*$. In addition,*

1. if $\phi(\omega) < 1$ for any ω, then $\alpha^* = 1$,

2. *if there is ω with $\phi(\omega) > 1$ and if $\bar{\phi} = \sum_w \pi(w)\phi(w) < 1$, then $0 < \alpha^* < 1$,*

3. *if $\bar{\phi} > 1$, then $\alpha^* = 0$.*

The REE is stable if and only $\alpha < \alpha^*$, i.e., the proportion of informed agents is low enough. Information revealed to some uninformed agents can only destabilize the REE. The sensitivity of the price to the forecasts of the α informed agents is related to $\phi(\omega)$, and the sensitivity of the price to the forecasts of the $(1 - \alpha)$ uninformed agents is related to $\sum_w \pi(w)\phi(w)$. It follows that the presence of informed agents is destabilizing in a state ω such that $\phi(\omega) > \sum_w \pi(w)\phi(w)$. As there is necessarily one such state, the presence of many informed agents tends to make the REE less stable. An intuition in line with Proposition 1 stems from the sensitivity of individual forecasts to others' behavior. When an uninformed agent expects the aggregate price forecast to change in some state, the adjustment in his own price forecast will be weighted by the probability of that state occurring. For this reason, his forecasting behavior is less sensitive to others' forecasts than the behavior of an informed agent. The uninformed agent's behavior is consequently easier to predict, which favors stability.

4.3 A discontinuity property

A version of the 'uncertainty principle' (Grandmont, 1998) directly follows from Proposition 3.

Corollary 5. *'Uncertainty principle.' For α close enough to 1, the REE is unstable if there is at least one state ω' such that $\phi(\omega') > 1$.*

Assume that the true underlying state ω is such that $\phi(\omega) < 1$. Under complete information, the REE is stable. Introducing a few uninformed agents yields a sudden change in the stability of the REE. As far as there is a state ω' such that $\phi(\omega') > 1$, the REE becomes unstable, and we are back to the characterization of rationalizable prices given in Proposition 2. This discontinuity follows from the contagion argument, which does not operate under complete information but is of importance in the presence of uninformed agents. The fact that there is a state in which agents would not succeed in predicting the correct price in the absence of uninformed agents implies that uninformed agents fail to single out one price forecast. This prevents stability.

Corollary 5 is reminiscent of the 'uncertainty principle' advocated by Grandmont (1998). According to this principle, adaptive learning dynamics diverge when agents are ready to use a large range of regularities, including divergent trends. In our setup, if uninformed agents believe it is possible that the true state may be a state in which the REE would be unstable under complete information, then the REE is unstable.

The discontinuity is also reminiscent of Morris and Shin's (1998) insights, in the sense that small informational asymmetries yield large qualitative changes.

In Morris and Shin (1998), this change concerns the number of equilibria: asymmetric information implies a unique REE. In our setup, this change concerns the number of rationalizable solutions: informational asymmetry creates a multiplicity of rationalizable solutions, whereas there is always a unique REE.

Our model generalizes Morris and Shin (1998), which is restricted to the situation where $\phi(\omega) = \phi \in [0, 1)$ (see Example 2). By Proposition 3, the REE is then always stable. However, the sensitivity ϕ can be made greater than 1 if the distribution of private signals is well chosen; in particular, private signals need to be weakly correlated. Our setup relaxes the CK assumption of ϕ by introducing uncertainty to, e.g., the distribution of private signals or to the size of the population of agents who may attack the currency. In this extended setup, Proposition 3 applies: informational asymmetry on ω sometimes creates multiple rationalizable solutions and REE instability.

It may be surprising that the likelihood of the occurrence of states in which the sensitivity to beliefs is greater than 1 does not matter in Corollary 5. Instability of the REE is obtained even if these states occur with a low probability. The likelihood of the occurrence of these states in fact influences rationalizable prices. Our next result qualifies Corollary 5.

Proposition 6. *Consider an unstable REE. Assume that $\pi(\omega)$ tend to 0 for every ω such that $\phi(\omega) > 1$.*

1. *For all ω such that $P^{\infty}(\omega) \subsetneq P^0$, $P^{\infty}(\omega)$ converges towards $\{p^*(\omega)\}$.*
2. *For all other states, $P^{\infty}(\omega)$ is P^0.*

Assume that the true underlying state is ω such that $\phi(\omega) < 1$. The REE is stable under complete information but loses its stability under the conditions given in Corollary 5. Nevertheless, if the probability of occurrence of all 'unstable' states (states with $\phi(w) < 1$) is small, the interval of possible prices $P^{\infty}(\omega)$ only comprises prices close to the equilibrium price $p^*(\omega)$. Volatility is therefore contained.

5 Discussion and extensions

5.1 Informational efficiency of the price

The rational expectations literature emphasizes the idea that the equilibrium price, once made public, reveals the underlying state of nature, provided that REE prices differ across states. This is the property of 'informational efficiency.' This concept easily generalizes to rationalizable prices: a price p reveals the state ω whenever $p \in P^{\infty}(\omega)$ and $p \notin P^{\infty}(\omega')$ for any $\omega' \neq \omega$. Informational efficiency of the price is obtained whenever any rationalizable price reveals the state, that is, $P^{\infty}(\omega)$ and $P^{\infty}(\omega')$ do not intersect for any $\omega' \neq \omega$.

Proposition 7. *Assume that the REE prices differ across states. Informational efficiency is obtained if and only if the REE is stable.*

The argument is straightforward. When the REE is stable, the sets $P^{\infty}(\omega)$ reduce to the REE prices and reveal the state. When the REE is unstable, it follows from Proposition 2 that there is a state ω_0 with $P^{\infty}(\omega_0) = P^0$. As every $P^{\infty}(\omega)$ is included in P^0, informational efficiency cannot be obtained.

Some rationalizable prices may, nevertheless, be fully revealing when the REE is unstable. Such prices exist only if there is a unique state ω_0 with $P^{\infty}(\omega_0) = P^0$. Indeed, in this configuration, every price in P^0, but not in any $P^{\infty}(\omega)$ for all $\omega \neq \omega_0$, reveals ω_0. No other fully revealing price exists. In particular, no price reveals a state ω with $\phi(\omega) < 1$.

Interpreting this result suggests that, when the REE is unstable, informational asymmetry should persist over time if the actual state ω is such that $\phi(\omega) < 1$. Indeed, as the price cannot reveal ω, there is no way uninformed agents can learn what is the actual state. Therefore, instability should persist over time, and there is no reason why the price should be equal to its equilibrium value. This scenario contrasts with what would happen if ω were CK, as the REE price is stable in this complete information case. In summation, the initial belief that $\phi(\omega) > 1$ with positive probability creates instability, and such a belief can never be rejected by public observation.

When the REE is unstable, and the actual state is such that $\phi(\omega) > 1$, agents may sometimes discover the true state of nature by using information revealed through price, but of course, this never implies stability of the REE.

5.2 Higher order uncertainty

Thus far, uninformed agents had used a common prior distribution of states, and this fact was CK. It may appear difficult to justify this assumption in a framework that otherwise stipulates a high level of ignorance. We now consider a case where every agent is uncertain about the distribution of the states of nature used by others. Let $(\pi_i(\omega))$ be the distribution used by an uninformed agent i. The aggregate forecast becomes

$$\int_0^{\alpha} p_i^e(\omega) + \int_{\alpha}^{1} \left(\sum_{w=1}^{\Omega} \pi_i(w)\, p_i^e(w) \right) di.$$

We define stability in this extended framework as we have done previously. For every ω, we define iteratively a sequence of sets of prices $P^{\tau}(\omega)$. For every ω, $P^0(\omega)$ is simply P^0, and $P^{\tau}(\omega)$ is defined using the sets $P^{\tau-1}(\omega)$ as follows. Consider that all the individual price forecasts $p_i^e(w)$ belong to $P^{\tau-1}(w) = \left[p_{\text{inf}}^{\tau-1}(w), p_{\text{sup}}^{\tau-1}(w)\right]$ for every w. The aggregate price forecast in state ω then lies in

$$\left[\alpha p_{\text{inf}}^{\tau}(\omega) + (1-\alpha) \inf_{w} p_{\text{inf}}^{\tau}(w), \alpha p_{\text{sup}}^{\tau}(\omega) + (1-\alpha) \sup_{w} p_{\text{sup}}^{\tau}(w)\right].$$

Making a more precise statement requires some knowledge about the prior distributions π_i used by uninformed agents. We assume that no agent has such knowledge. In particular, all the probability distributions $(\pi_i(\omega)$ can be used by uninformed agents. It follows that every agent only knows that the price in state ω lies in an interval $P^{\tau}(\omega) = \left[p_{\text{inf}}^{\tau}(\omega), p_{\text{sup}}^{\tau}(\omega)\right]$ where $p_{\text{inf}}^{\tau}(\omega)$ is the price associated with the smallest possible aggregate forecast:

$$p_{\text{inf}}^{\tau}(\omega) \equiv \phi(\omega) \left(\alpha p_{\text{inf}}^{\tau-1}(\omega) + (1-\alpha) \inf_{w} p_{\text{inf}}^{\tau-1}(w)\right) + \eta(\omega), \tag{8}$$

and $p_{\sup}^{\tau}(\omega)$ is the price associated with the largest possible aggregate forecast:

$$p_{\sup}^{\tau}(\omega) \equiv \phi(\omega)\left(\alpha p_{\sup}^{\tau-1}(\omega) + (1-\alpha)\sup_{w} p_{\sup}^{\tau-1}(w)\right) + \eta(\omega). \tag{9}$$

The sequence $P^{\tau}(\omega)$ converges to a limit set $P^{\infty}(\omega) = \left[p_{\inf}^{\infty}(\omega), p_{\sup}^{\infty}(\omega)\right]$ for every ω, where the values $p_{\inf}^{\infty}(\omega)$ and $p_{\sup}^{\infty}(\omega)$ are fixed points of (8) and (9). It is not possible for $P^{\infty}(\omega)$ to reduce to a single price. This means that higher order uncertainty prevents agents from discovering the REE. Agents may at most discover an interval of possible prices $P^{\infty}(\omega) \subsetneq P^{0}$ in every state ω. A rise in the proportion of informed agents has two types of effects. First, it influences the convergence of the process of elimination of dominated strategies. Second, it affects the size of the intervals of possible prices in case of stability, that is, whenever $P^{\infty}(\omega) \subsetneq P^{0}$ in every state ω. In light of Proposition 3 and Corollary 4, more precise information may not only destabilize the learning process but may also narrow the set of possible solutions in case of stability.

5.3 Sunspots and stability

Consider, finally, a stochastic sunspot variable that can take Σ values ($S = 1, \ldots, \Sigma$), not correlated with fundamentals. Assume that its actual value is not known when agents form their forecasts. Every agent i observes a private signal $s_i = 1, \ldots, \Sigma$ imperfectly correlated with S. Conditionally based on S, private signals are independently and identically distributed across agents, and the probability $\Pr(s_i \mid S)$ that i observes s_i in sunspot event S is independent of i. Thus, in sunspot event S, there are $\Pr(s \mid S)$ agents who observe the signal s ($s = 1, \ldots, \Sigma$).

Suppose that all the agents expect the price $p^e(\omega, S)$ to arise if the state of fundamentals is ω and the sunspot is S. In state (ω, S), there are $\alpha \Pr(s \mid S)$ informed agents whose price forecast is

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) p^e(\omega, S')$$

for any s. There are also $(1-\alpha)\Pr(s \mid S)$ uninformed agents who expect

$$\sum_{S'=1}^{\Sigma} \Pr(S' \mid s) \sum_{w=1}^{\Omega} \pi(w)\, p^e(w, S').$$

Let

$$\mu(S'|S) = \sum_{s=1}^{\Sigma} \Pr(s \mid S) \Pr(S' \mid s)$$

be the average probability (across agents) of sunspot S' if the actual sunspot is S. The aggregate price forecast $P^e(\omega, S)$ is expressed as

$$\sum_{S'=1}^{\Sigma} \mu(S'|S) \left[\alpha p^e(\omega, S') + (1-\alpha) \sum_{w=1}^{\Omega} \pi(w)\, p^e(w, S')\right], \tag{10}$$

and the actual price $p(\omega, S)$, determined by (1) in state (ω, S), is such that

$$p(\omega, S) = \phi(\omega)\, P^e(\omega, S) + \eta(\omega). \tag{11}$$

A REE is now a vector of $\Omega\Sigma$ prices $(p^*(1,1),\ldots,p^*(\Omega,\Sigma))$ such that $p^e(\omega,S) = p(\omega,S) = p^*(\omega,S)$ for every (ω,S) in (10) and (11). The 'fundamental' REE is obtained when $p^*(\omega,S)$ is independent of S. Otherwise, sunspots matter and the REE is a 'sunspot' equilibrium.

The following result gives conditions for the existence of a sunspot REE.

Proposition 8. *There exists a sunspot REE if and only if the fundamental REE is unstable in (6).*

In our linear setup, the stability of the fundamental REE is still ruled by Proposition 3 and Corollary 4. Hence, both results also give necessary and sufficient conditions nedded for the sunspot REE to exist. This highlights another destabilizing effect caused by informed agents: sunspots only matter when many agents are informed about the true state of nature.

6 Appendix

Examples with an uncertain sensitivity to beliefs

Example 1'. The Muth model with an uncertain aggregate demand. Let the aggregate demand in state ω be $b(\omega)-a(\omega)p$. The expected profit of an informed farmer i is $p_i^e(\omega)q - q^2/\sigma$ and his supply is $q_i(\omega) = \sigma p_i^e(\omega)$. The expected profit of an uninformed farmer is $\sum \pi(w)p_i^e(w)q - q^2/\sigma$, so that his production is $q_i = \sigma \sum \pi(w)p_i^e(w)$. In equilibrium, the actual price $p(\omega)$ in state ω is such that

$$\sigma\left(\int_0^\alpha p_i^e(\omega)di + \int_\alpha^1 \sum \pi(w)p_i^e(w)di\right) = -a(\omega)p(\omega) + b(\omega).$$

Example 2'. Investment game with an uncertain size of the population of investors. Consider the setup of Example 2. Assume now that the size of the population of investors is $n(\omega)$ in state ω, with $\sum \pi(w)n(w) = 1$. An investor who knows the state of nature ω expects the proportion of agents who decide to invest to be equal to

$$n(\omega)\left(\int_0^\alpha (1 - F(k_j(\omega) \mid x))dj + \int_\alpha^1 (1 - F(k_j \mid x))dj\right).$$

Thus the best response of an informed agent is to invest if and only if his private signal is above

$$k_i(\omega) = n(\omega)\left(\int_0^\alpha F(k_j(\omega) \mid k_i(\omega))dj + \int_\alpha^1 F(k_j \mid k_i(\omega))dj\right) + 1 - n(\omega).$$

The best response of an uninformed agent is

$$k_i = \sum_w \pi(w)n(\omega)\left(\int_0^\alpha F(k_j(\omega) \mid k_i(\omega))dj + \int_\alpha^1 F(k_j \mid k_i(\omega))dj\right).$$

The setup is linear in the case of uniform conditional distribution: given x, y is uniform on $[x-\varepsilon, x+\varepsilon]$. Indeed, assuming that the thresholds are close enough to each other, the best response function of the informed agent rewrites

$$k_i(\omega) = \frac{n(\omega)}{2\varepsilon + n(\omega)} \left(\int_0^\alpha k_j(\omega)\, dj + \int_\alpha^1 k_j dj \right) + \frac{2-n(\omega)}{2\varepsilon + n(\omega)} \varepsilon.$$

This coincides with the reduced form (5), with the variable $k(\omega)$ replacing $p(\omega)$. It remains to check that the best response of an uninformed agent is an average of the best responses of informed agents over the states of nature. Indeed, using the adjusted probabilities

$$\hat{\pi}(\omega) = \pi(\omega) \frac{n(\omega) + 2\varepsilon}{1+2\varepsilon},$$

the best response of an uninformed agent writes

$$k_i = \sum_w \hat{\pi}(w)\, k_i(w) - \frac{1}{1+2\varepsilon}.$$

This fits our guessing game (the additive constant in the expression of k_i can be seen as part of the constant term $\eta(\omega)$ in (5)).

Proof of Proposition 2

1. Since the equilibrium is unstable, there is a state ω such that $\{p^*(\omega)\} \subsetneq P^\infty(\omega)$. The set of rationalizable price forecasts of uninformed agents cannot be reduced to a single element. In any given state the set of rationalizable prices is determined by the aggregate price forecast in that state, which depends on the forecasts of uninformed agents. Thus in any given state the aggregate price forecast cannot reduce to a single point.

2. Consider the minimum rationalizable prices $(p^\infty_{\text{inf}}(\omega))_\omega$. For ω such that $\alpha\phi(\omega) > 1$, we show that $p^\infty_{\text{inf}}(\omega) = p^0_{\text{inf}}$. To this purpose, we show that, when everyone expects $(p^\infty_{\text{inf}}(\omega))_\omega$, we have

$$p^0_{\text{inf}} \geq \alpha\phi(\omega)\, p^0_{\text{inf}} + (1-\alpha)\, \phi(\omega) \sum \pi(w)\, p^\infty_{\text{inf}}(w) + \eta(\omega), \qquad (12)$$

which means that p^0_{inf} is the actual price in state ω (that is: $p^0_{\text{inf}} = p^\infty_{\text{inf}}(\omega)$). Recall the fixed point relation characterizing the equilibrium $(p^*(\omega))_\omega$

$$p^*(\omega) = \alpha\phi(\omega)\, p^*(\omega) + (1-\alpha)\, \phi(\omega) \sum \pi(w)\, p^*(w) + \eta(\omega).$$

Substracting this equality to (12) gives

$$\Delta p(\omega) > \alpha\phi(\omega)\, \Delta p(\omega) + (1-\alpha)\, \phi(\omega) \sum \pi(w)\, \Delta p(w),$$

where $\Delta p(\omega) = p^\infty_{\text{inf}}(\omega) - p^*(\omega) \leq 0$. This rewrites

$$(1-\alpha\phi(\omega))\, \Delta p(\omega) \geq (1-\alpha)\, \phi(\omega) \sum \pi(w)\, \Delta p(w),$$

which holds true as

$$(1-\alpha\phi(\omega))\,\Delta p(\omega) \geq 0 \geq (1-\alpha)\,\phi(\omega)\sum \pi(w)\,\Delta p(w)\,.$$

The same argument shows that $p_{\sup}^{\infty}(\omega) = p_{\sup}^{0}$ for every ω such that $\alpha\phi(\omega) > 1$.

3. The third item follows from the first step of the iterative process. By assumption, the equilibrium price $\eta(\omega)/(1-\phi(\omega))$ under complete information belongs to P^0. From (6), at the first step of the process, we have:
$$p_{\inf}^{1}(\omega) = \max\left(p_{\inf}^{0}, \phi(\omega)\,p_{\inf}^{0} + \eta(\omega)\right).$$
Since $p_{\inf}^{0} < \eta(\omega)/(1-\phi(\omega)$ and $\phi(\omega) < 1$, we have $p_{\inf}^{1}(\omega) > p_{\inf}^{0}$. By definition, the map $R_\omega(P)$ cannot be increasing with τ. It follows that $p_{\inf}^{\infty}(\omega) \geq p_{\inf}^{1}(\omega) > p_{\inf}^{0}$. The same argument shows that
$$p_{\sup}^{1}(\omega) = \min\left(p_{\sup}^{0}, \phi(\omega)\,p_{\sup}^{0} + \eta(\omega)\right) < p_{\sup}^{0},$$
so that $p_{\sup}^{\infty}(\omega) \leq p_{\sup}^{1}(\omega) < p_{\sup}^{0}$. This shows that $P^{\infty}(\omega)$ is a strict subset of P^0.

Proof of Proposition 3

Consider, e.g., the Ω equations in (6) corresponding to the lowest bounds $P_{\inf}^{\tau}(\omega)$ of $P^{\tau}(\omega)$. They can be rewritten in matrix form $\mathbf{p}_{\inf}^{\tau+1} = \mathbf{M}\mathbf{p}_{\inf}^{\tau} + \boldsymbol{\eta}$, where $\mathbf{p}_{\inf}^{\tau}$ is the $\Omega \times 1$ vector $(P_{\inf}^{\tau}(1), \ldots, P_{\inf}^{\tau}(\Omega))$, $\boldsymbol{\eta}$ is the $\Omega \times 1$ vector $(\eta(1), ..., \eta(\Omega))$, and $\mathbf{M}$ is the $\Omega \times \Omega$ matrix $\alpha\mathbf{\Phi} + (1-\alpha)\,\mathbf{\Phi\Pi}$ (with $\mathbf{\Phi}$ the diagonal $\Omega \times \Omega$ matrix whose $\omega\omega$th entry is $\phi(\omega)$, and $\mathbf{\Pi}$ the $\Omega \times \Omega$ stochastic matrix whose $\omega\omega'$th entry is $\pi(\omega')$). The REE is stable if and only if the spectral radius $\rho(\mathbf{M})$ of $\mathbf{M}$ is less than 1. The proof now hinges on the fact that for any $\Omega \times \Omega$ positive matrix $\mathbf{M}$, and any $\Omega \times 1$ vector $\mathbf{x} = (x_\omega)$ with every $x_\omega > 0$, we have
$$\min_{\omega} \frac{(\mathbf{Mx})_\omega}{x_\omega} \leq \rho(\mathbf{M}) \leq \max_{\omega} \frac{(\mathbf{Mx})_\omega}{x_\omega},$$
where $(\mathbf{Mx})_\omega$ stands for the ωth component of the $\Omega \times 1$ vector $\mathbf{Mx}$ (see Lemma 3.1.2. in Bapat and Raghavan (1997)). Let

$$Q(\mathbf{x},\omega) = \frac{(\mathbf{Mx})_\omega}{x_\omega} = \phi(\omega)\left[\alpha + (1-\alpha)\,\frac{1}{x_\omega}\sum_{w=1}^{\Omega}\pi(w)\,x_w\right],$$

for any ω. Assume first that $\alpha\phi(\omega) > 1$ for some ω, e.g. $\omega = \Omega$. Then, consider the vector $\mathbf{x} = (\varepsilon, \ldots, \varepsilon, 1)'$ where $\varepsilon > 0$. When ε tends toward 0, $Q(\mathbf{x},\omega)$ tends to $(+\infty)$ for every $\omega < \Omega$, and $Q(\mathbf{x},\Omega) \geq \alpha\phi(\Omega) > 1$. Hence, $\min_\omega Q(\mathbf{x},\omega) > 1$ for ε small enough, and so $\rho(\mathbf{M}) > 1$: the REE is unstable if $\alpha\phi(\omega) > 1$ for some ω. If, on the contrary, $\alpha\phi(\omega) < 1$ for any ω, then define

$$E = \sum_{w=1}^{\Omega}\pi(w)\,\frac{(1-\alpha)\,\phi(w)}{1-\alpha\phi(w)}.$$

Consider the $\Omega \times 1$ positive vector $\mathbf{x}$ whose ωth component is

$$x_\omega = \frac{1}{E} \frac{(1-\alpha)\,\phi(\omega)}{1-\alpha\phi(\omega)}.$$

If $E \geq 1$, then $Q(\mathbf{x}, \omega) > 1$ for any ω, so that $\min_\omega Q(\mathbf{x}, \omega) \geq 1$, and the REE is unstable. If, on the contrary, $E < 1$, then $Q(\mathbf{x}, \omega) < 1$ for any ω, so that $\max_\omega Q(\mathbf{x}, \omega) < 1$, and the REE is stable.

Proof of Corollary 4

1. Assume first that $\phi(\omega) < 1$ for any $\omega = 1, \ldots, \Omega$. Then, $\alpha\phi(\omega) < 1$ and $(1-\alpha)\,\phi(\omega) / (1-\alpha\phi(\omega)) < 1$ for any ω. By Proposition 3, the REE is stable.

2. Let now $\inf_\omega \phi(\omega) < 1 < \sup_\omega \phi(\omega)$. If $\alpha > 1/\sup_\omega \phi(\omega)$, the REE is unstable, by Proposition 3. If $\alpha \leq 1/\sup_\omega \phi(\omega)$, then $\alpha\phi(\omega) < 1$ for every ω, and the REE is stable if and only if (7) is met. Let

$$F(\alpha) = \sum_{w=1}^{\Omega} \pi(w) \frac{\phi(w)}{1-\alpha\phi(w)} - \frac{1}{(1-\alpha)} \tag{13}$$

   Since $F(\cdot)$ is a continuous and increasing function of α on the interval $[0, 1/\sup_\omega \phi(\omega)]$, with $F'(\alpha) > 0$ whatever α is, there is at most one value α such that $F(\alpha) = 0$ on this interval. Observe now that $F(0) = \bar{\phi} - 1$, and $F(\alpha)$ tends to $+\infty$ when α tends to $1/\sup_\omega \phi(\omega)$ from below. If, on the one hand, $\bar{\phi} \geq 1$, then $F(\alpha) \geq F(0) > 0$ for any $\alpha \in [0, 1/\sup_\omega \phi(\omega)]$, and the stability condition (7) is never satisfied. If, on the other hand, $\bar{\phi} < 1$, then there exists a unique solution α^* ($\alpha^* > 0$) to $F(\alpha) = 0$ in $[0, 1/\sup_\omega \phi(\omega)]$. The condition $F(\alpha) < 0$, i.e. the stability condition (7), is equivalent to $\alpha < \alpha^*$. Since $F(\alpha^*) = 0$ implicitly defines α^* as a function $(\phi(1), \ldots, \phi(\Omega))$, and since $F(\cdot)$ increases in every $\phi(\omega)$, α^* decreases in every $\phi(\omega)$.

3. Let $\bar{\phi} > 1$. We know that $F(\alpha) > 0$ for any $\alpha \in [0, 1/\sup_\omega \phi(\omega)]$. As a result, the stability condition (7) is never satisfied.

Proof of Proposition 6

1. Consider a state ω such that $P^\infty(\omega) \subsetneq P^0$. In particular, $p^\infty_{\inf}(\omega) > p^0_{\inf}$. From (6),

$$p^\infty_{\inf}(\omega) = \phi(\omega) \left(\alpha p^\infty_{\inf}(\omega) + (1-\alpha) \sum_{w/\phi(w)<1} \pi(w) p^\infty_{\inf}(w) \right)$$

$$+\phi(\omega)(1-\alpha) \sum_{w/\phi(w)>1} \pi(w) p^\infty_{\inf}(w) + \eta(\omega).$$

   When every $\pi(w)$ such that $\phi(w) > 1$ tends to 0, the last sum also tends to 0 since every $p^\infty_{\inf}(w)$ is in a compact set ($p^\infty_{\inf}(w) \in P_0$). Hence, when every $\pi(w)$ such that $\phi(w) > 1$ tends to 0, the values $p^{\infty}_{\inf}(\omega)$ for the states ω with $\phi(\omega) < 1$ tends to the solution of

$$p^{\infty}_{\inf}(\omega) = \phi(\omega) \left(\alpha p^{\infty}_{\inf}(\omega) + (1-\alpha) \sum_{w/\phi(w)<1} \pi(w) p^{\infty}_{\inf}(w) \right) + \eta(\omega).$$

This coincides with (4), and thus the limit value of $p^{\infty}_{\inf}(\omega)$ is the REE price for every ω such that $P^{\infty}(\omega) \subsetneq P^{0}$ when every $\pi(w)$ such that $\phi(w) > 1$ is close to 0. An analogous argument applies to $p^{\infty}_{\sup}$.

2. See Item 2 of Proposition 2.

Proof of Proposition 8

Let us rewrite conditions (11) in matrix form. To this aim, let $\mathbf{p}(S)$ be the $\Omega \times 1$ vector whose ωth component is $p(\omega, S)$, and $\mathbf{p}$ be the $\Omega\Sigma \times 1$ vector $(\mathbf{p}(1), \ldots, \mathbf{p}(\Sigma))$. Let $\mathbf{S}$ be the $\Sigma \times \Sigma$ stochastic matrix whose SS'th entry is $\mu(S', S)$. Then, with $\mathbf{M}$ defined in Proposition 3, a REE is a vector $\mathbf{p}$ such that

$$\mathbf{p} = (\mathbf{M} \otimes \mathbf{S})\, \mathbf{p} + \mathbf{1}_{\Sigma} \otimes \boldsymbol{\eta}, \tag{14}$$

where the symbol $\otimes$ stands for the Kronecker product. Let $e(S)$ be the Sth eigenvalue of $\mathbf{S}$, with $e(S) \in [-1, 1]$ since $\mathbf{S}$ is a stochastic matrix. Let $\mu(\omega)$ be the ωth eigenvalue of $\mathbf{M}$. Then, the $\Omega\Sigma$ eigenvalues of $\mathbf{M} \otimes \mathbf{S}$ are $e(S)\mu(\omega)$ for any pair (ω, S). If $\rho(\mathbf{M}) < 1$, then all the eigenvalues of $\mathbf{M} \otimes \mathbf{S}$ have moduli less than 1, and so $\mathbf{M} \otimes \mathbf{S} - \mathbf{I}_{2\Omega}$ is invertible and there is a unique REE. If $\rho(\mathbf{M}) \geq 1$, there exist stochastic matrices such that $e(S) = 1/\rho(\mathbf{M})$ for some S. In this case, the matrix $\mathbf{M} \otimes \mathbf{S}$ has an eigenvalue equal to 1, and there are infinitely many $\mathbf{p}$ solution to (14), i.e. infinitely many sunspot REE and the fundamental REE.

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